Fast quantum integer multiplication without ancillas

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Multiplication on quantum computers

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... **with as few gates and qubits as possible.**

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$$\mathcal{U}_{q\times q}\left|x\right\rangle\left|y\right\rangle\left|w\right\rangle = \left|x\right\rangle\left|y\right\rangle\left|w + xy\right\rangle$$

$$\mathcal{U}_{c\times q}(a)\left|x\right\rangle\left|w\right\rangle = \left|x\right\rangle\left|w + ax\right\rangle$$

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Karatsuba (Gidney ’19):

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- Ancillas: $\mathcal{O}(n)$

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Karatsuba (Gidney '19):

- Gates: $\mathcal{O}(n^{1.58\cdots})$
- Ancillas: $\mathcal{O}(n)$

This work:

- Gates: $\mathcal{O}(n^{1+\epsilon})$ for any $\epsilon > 0$
- Ancillas: 0

Results (spoilers): in practice

Cost to multiply a 2048-bit quantum register by a 2048-bit classical value:

[1] Gidney '19, "Windowed quantum arithmetic"

Algorithm	Asymptotic scaling	Gate count (millions)			Ancillas
		Toffoli	CR_ϕ	H,X,CNOT	
Schoolbook [1]	$\mathcal{O}(n^2)$	6.4	—	38	1*
Karatsuba [1]	$\mathcal{O}(n^{1.58})$	5.6	—	34	12730
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This work (standard QFT)	$\mathcal{O}(n^{1.29})$**	**0.6**	**0.3**	**1.9**	**79**
This work (phase gradient QFT)	$\mathcal{O}(n^{1.29})$**	**0.9**	**0.1**	**3.2**	**80**

Plan

1. Background and core algorithm (slides)

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2. Practical considerations and optimizations (choose your own adventure)

Background: schoolbook multiplication

The "schoolbook" method: $xy = \sum_{ij}(2^i x_i)(2^j y_j) = \sum_{ij} 2^{i+j} x_i y_j$

$$
\begin{array}{cccccccc}
 & & & & 1 & 1 & 0 & 1 \\
 & & & \times & 1 & 0 & 1 & 0 \\
\hline
 & & & & 1 & 0 & 1 & 0 \\
 & & 1 & 0 & 1 & 0 & & \\
+ & 1 & 0 & 1 & 0 & & & \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
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Running time: $\mathcal{O}(n^2)$ operations

Background: schoolbook multiplication

Given two n-bit numbers x and y, what if we use base $b = 2^{n/2}$?

$$
\begin{array}{rcccc}
 & & & x_1 & x_0 \\
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 & & & & x_0 y_0 \\
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$$xy = x_1 y_1 b^2 + x_0 y_1 b + x_1 y_0 b + x_0 y_0$$

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$$xy = x_1 y_1 b^2 + x_0 y_1 b + x_1 y_0 b + x_0 y_0$$

Time remains $\mathcal{O}(n^2)$, because $4(n/2)^2 = n^2$

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$$xy = x_1y_1b^2 + (x_0y_1 + x_1y_0)b + x_0y_0$$

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Observation: $x_0 y_1 + x_1 y_0 = (x_1 + x_0)(y_1 + y_0) - x_1 y_1 - x_0 y_0$

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Can compute xy with only three multiplications of size $\log b = n/2$:

1. x_1y_1
2. x_0y_0
3. $(x_1 + x_0)(y_1 + y_0)$

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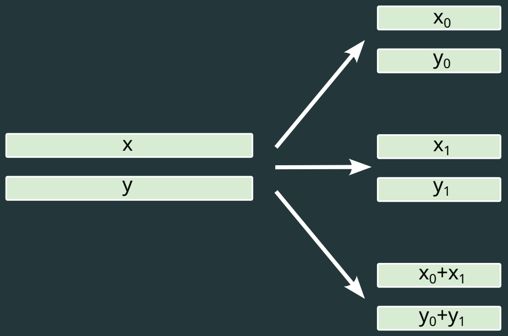
1. $x_1 y_1$
2. $x_0 y_0$
3. $(x_1 + x_0)(y_1 + y_0)$

Computational cost: $3(n/2)^2 = \frac{3}{4} n^2 = \mathcal{O}(n^2)$

Background: Karatsuba multiplication



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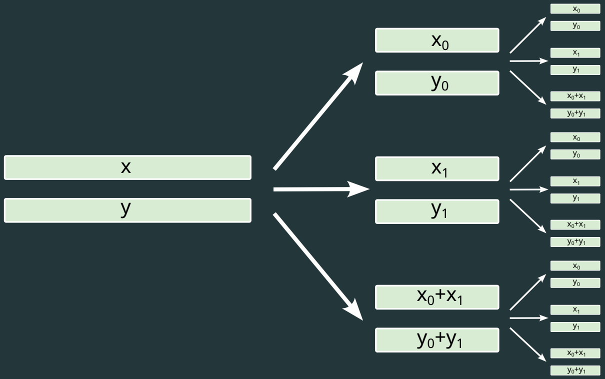

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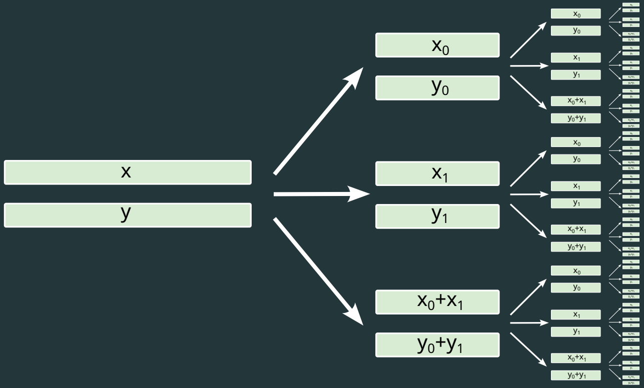

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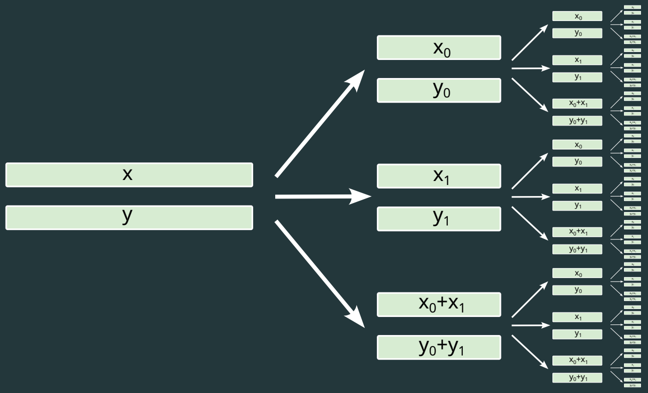

Depth: $d = \log_2 n$

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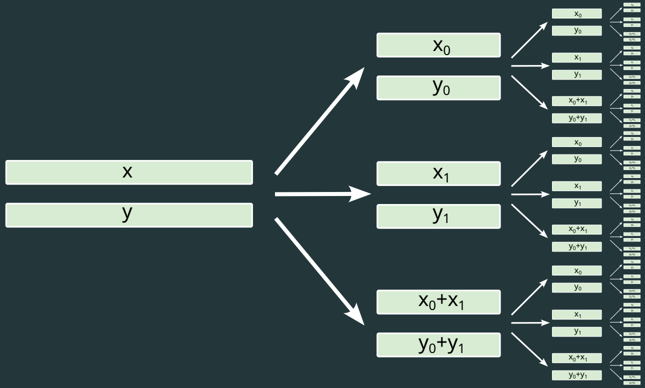

Depth: $d = \log_2 n$

Operations: 3^d

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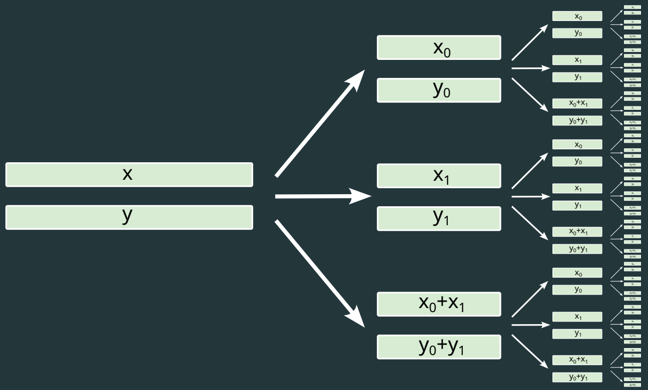

Depth: $d = \log_2 n$

Operations: 3^d

Cost: $\mathcal{O}(n^{\log_2 3}) = \mathcal{O}(n^{1.58\cdots})$

A fundamentally quantum way of doing arithmetic

[Draper '04]: Arithmetic in Fourier space

$$|xy\rangle = \mathsf{QFT}^{-1} \sum_{z} \exp\left(\frac{2\pi i xyz}{2^n}\right) |z\rangle$$

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1) Generate $|x\rangle\,|y\rangle \sum_z |z\rangle$, 2) apply a phase rotation of $\exp\left(\frac{2\pi i xyz}{2^n}\right)$

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1) Generate $|x\rangle\,|y\rangle \sum_z |z\rangle$, 2) apply a phase rotation of $\exp\left(\frac{2\pi i x y z}{2^n}\right)$, 3) apply QFT^{-1}

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$$\exp\left(\frac{2\pi i xyz}{2^n}\right) = \prod_{i,j,k} \exp\left(\frac{2\pi i 2^{i+j+k}}{2^n} x_i y_j z_k\right)$$

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x_i, y_j, z_k are binary values—apply phase only if they all are equal to 1!

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A series of CCR_ϕ gates between the bits of $|x\rangle$, $|y\rangle$, and $|z\rangle$!

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A modest improvement: classical-quantum multiplication $\mathcal{U}(a)\,|x\rangle\,|0\rangle = |x\rangle\,|ax\rangle$

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A modest improvement: classical-quantum multiplication $\mathcal{U}(a)\left|x\right\rangle\left|0\right\rangle = \left|x\right\rangle\left|ax\right\rangle$

$$\exp\left(\frac{2\pi i axz}{2^n}\right) = \prod_{i,j} \exp\left(\frac{2\pi i a 2^{i+j}}{2^n} x_i z_j\right)$$

Here: $\mathcal{O}(n^2)$ controlled phase rotations (matches Schoolbook algorithm)

Fast quantum multiplication

Main question: Can we combine fast multiplication with Fourier arithmetic to get the benefits of both?

Fast classical-quantum multiplication

Goal: $\mathcal{U}(a)\,|x\rangle\,|0\rangle = |x\rangle\,|ax\rangle$

Fast classical-quantum multiplication

Goal: Apply phase $\exp\left(\frac{2\pi i a}{2^n} xz\right)$; x and z are quantum

Fast classical-quantum multiplication

Goal: Implement PhaseProduct$(\phi)\left|x\right\rangle\left|z\right\rangle = \exp\left(i\phi xz\right)\left|x\right\rangle\left|z\right\rangle$

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We want to split the phase ϕxz into the sum of many phases, which are easy to implement.

Fast classical-quantum multiplication

Goal: Implement PhaseProduct(ϕ) $|x\rangle\,|z\rangle = \exp(i\phi xz)\,|x\rangle\,|z\rangle$

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Previously:

$$\exp(i\phi xz) = \prod_{i,j} \exp\left(i\phi 2^{i+j} x_i z_j\right)$$

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Karatsuba:

$$xz = 2^n x_1 z_1 + 2^{n/2}((x_0 + x_1)(z_0 + z_1) - x_0 z_0 - x_1 z_1) + x_0 z_0$$

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Plugging in Karatsuba:

$$\begin{aligned}\exp(i\phi xz) = &\exp\left(i\phi 2^n x_1 z_1\right)\\ &\cdot \exp\left(i\phi x_0 z_0\right)\\ &\cdot \exp\left(i\phi 2^{n/2}((x_0+x_1)(z_0+z_1) - x_0 z_0 - x_1 z_1)\right)\end{aligned}$$

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How are we supposed to reuse values in the *phase*?

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Re-ordering Karatsuba:

$$xz = (2^n - 2^{n/2})x_1z_1 + 2^{n/2}(x_0 + x_1)(z_0 + z_1) + (1 - 2^{n/2})x_0z_0$$

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Plugging in reordered Karatsuba:

$$\begin{aligned}\exp(i\phi xz) = &\exp\left(i\phi(2^n - 2^{n/2})x_1 z_1\right)\\ &\cdot \exp\left(i\phi(1 - 2^{n/2})x_0 z_0\right)\\ &\cdot \exp\left(i\phi 2^{n/2}(x_0 + x_1)(z_0 + z_1)\right)\end{aligned}$$

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Each of these has the same structure, but on half as many qubits $\rightarrow$ do it recursively!

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Recursion relation: $T(n) = 3T(n/2)$

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Recursion relation: $T(n) = 3T(n/2) \Rightarrow \mathcal{O}(n^{\log_2 3}) = \mathcal{O}(n^{1.58\cdots})$ gates!

How many qubits do we need?

Splitting registers $|x\rangle \to |x_1\rangle |x_0\rangle$ and $|z\rangle \to |z_1\rangle |z_0\rangle$, can immediately do

- $\exp(i\phi_1 x_1 z_1)$
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What about $\exp(i\phi_3(x_0 + x_1)(z_0 + z_1))$?

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Use quantum addition circuits.

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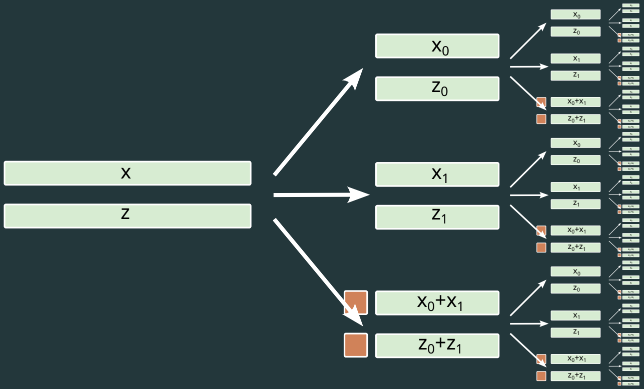
What about $\exp(i\phi_3 (x_0 + x_1)(z_0 + z_1))$?

Use quantum addition circuits.

But, **addition is reversible** $\to$ do it *in-place*! E.g. $|x_1\rangle |x_0\rangle \to |x_1\rangle |x_0 + x_1\rangle$

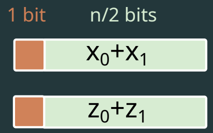
How many qubits do we need?

Total number of ancillas: $\mathcal{O}(\log n)$



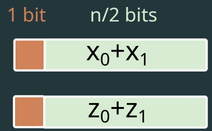
How many qubits do we need?

Total number of ancillas: $\mathcal{O}(\log n)$



How many qubits do we need?

Total number of ancillas: 2



Idea: “Shave off” the high bit before recursing

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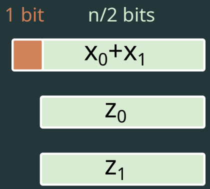
Total number of ancillas: 1



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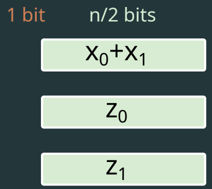

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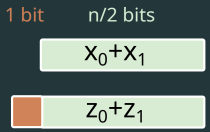

Idea: “Shave off” the high bit before recursing

How many qubits do we need?

Total number of ancillas: 1

1 bit	n/2 bits
	x_0+x_1
	z_0+z_1

Idea: “Shave off” the high bit before recursing

How many qubits do we need?

Total number of ancillas: 0

1 bit   n/2 bits

x_0+x_1

z_0+z_1

Idea: “Shave off” the high bit before recursing

Trick: Using dirty qubits, can reduce to zero!

Making it go faster

So far: $\mathcal{O}(n^{1.58})$ gates using zero ancillas

Making it go faster

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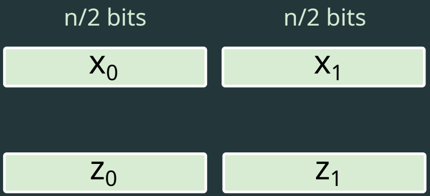
Can we make it go faster?

Generalizing: Toom-Cook multiplication

n/2 bits	n/2 bits
x_0	x_1
z_0	z_1



Let $b = 2^{n/2}$.

$$x = x_1 b + x_0$$
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$$x = \sum_{i=0}^{k-1} x_i b^i \qquad z = \sum_{i=0}^{k-1} z_i b^i$$

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$$\phi xz = \sum_{\ell=1}^{2k-1} \phi_\ell \left(\sum_i^k w_\ell^i x_i\right) \left(\sum_i^k w_\ell^i z_i\right)$$

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Let $k = 2$.

$w_1 = 0$

$$\sum_{i=0}^{k-1} w_1^i x_i = x_0 \tag{1}$$

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$$1/w_3 \sum_{i=0}^{k-1} w_3^i x_i = x_1 \tag{1}$$

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Complexity vs. k

This strategy yields asymptotic complexity $\mathcal{O}(n^{\log_k(2k-1)})$

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Algorithm	Gate count
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$k = 5$	$\mathcal{O}(n^{1.37\cdots})$
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Note: For quantum-quantum mult., get $\mathcal{O}(n^{\log_k(3k-2)})$

Thank you!

arXiv:2403.18006

Greg Kahanamoku-Meyer — gkm@mit.edu — https://gregkm.me/

I will be at Google QSS, including Friday resource estimation workshop!

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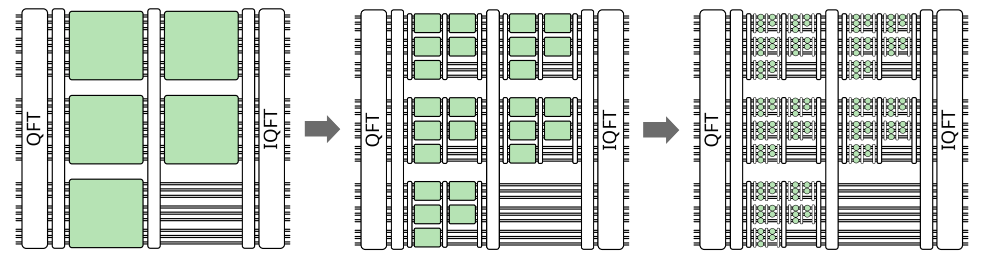
Further discussion:

- Circuit depth (new result: log-depth approx. QFT with few ancillas)
- Optimizing the base case, and implementing arbitrary phase rotations
- Optimizing choice of k
- Optimizing choice of w_ℓ and computation of linear combinations
- Modular arithmetic
- Dirty qubit construction

Backup

Circuit structure (depth and locality)



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1024 $CR_\phi \to 64$ R_ϕ plus ~ 2048 Toffoli

Fast exact quantum Fourier transform

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For any $m < n$, we may implement QFT_{2^n}:

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   - $|x\rangle$ is value of first m qubits
   - $|z\rangle$ is value of final $n - m$ qubits
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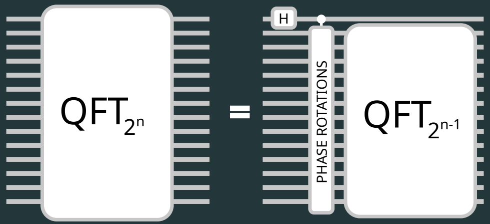

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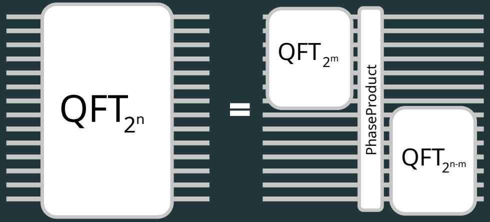

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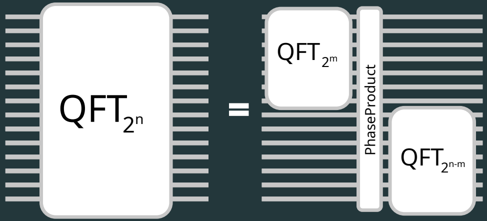
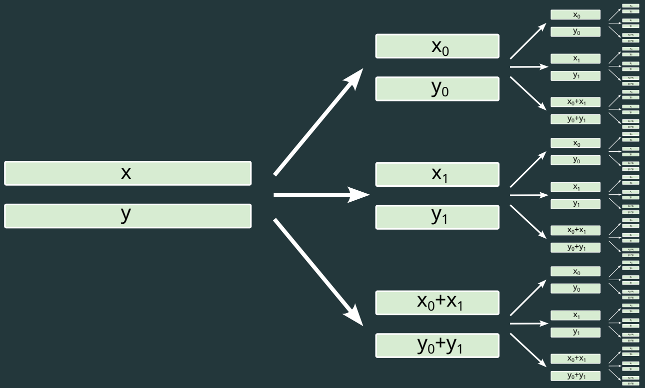


Immediately gives us sub-quadratic exact QFT using only 1 ancilla.

Depth considerations

Parallelization is natural.



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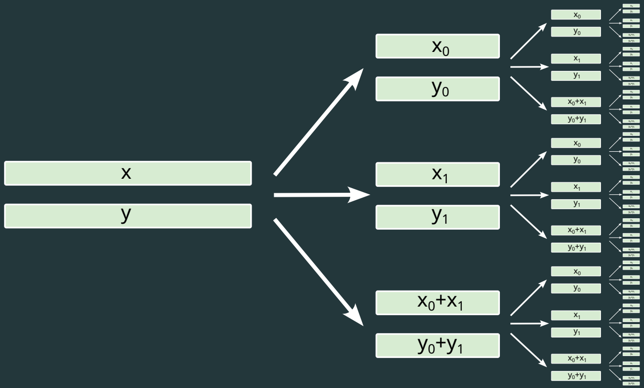
Parallelization is natural.

We have k sub-registers to work with—can do k sub-products in parallel.



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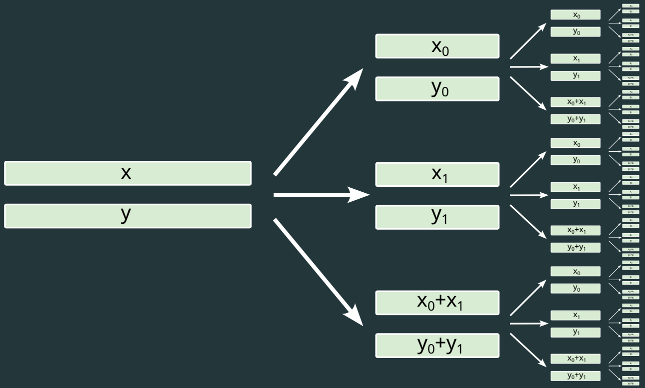

Depth: PhaseProduct in $\mathcal{O}(n^{\log_k 2})$ and PhaseTripleProduct in $\mathcal{O}(n^{\log_k 3})$ using a few more ancillas

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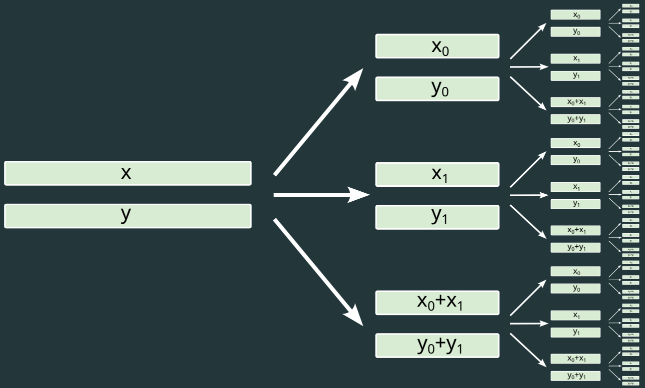

Challenge for multiply: How to do the QFT in sublinear depth with even $\mathcal{O}(n)$ ancillas?

So far: have been using phase

$$\exp\left(2\pi i \frac{xyz}{2^n}\right)$$

Modular arithmetic

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Output register requires $n + \mathcal{O}(\log(1/\epsilon))$ qubits

Fast classical-quantum multiplication: algorithm

PhaseProduct($\phi, |x\rangle, |z\rangle$)

Input: Quantum state $|x\rangle |z\rangle$, classical value ϕ

Output: Quantum state $\exp(i\phi xz) |x\rangle |z\rangle$

1. Split $|x\rangle$ and $|z\rangle$ in half, as $|x_1\rangle |x_0\rangle$ and $|z_1\rangle |z_0\rangle$
2. Apply PhaseProduct($(2^n - 2^{n/2})\phi, |x_1\rangle, |z_1\rangle$)
3. Apply PhaseProduct($(1 - 2^{n/2})\phi, |x_0\rangle, |z_0\rangle$)
4. Add $|x_1\rangle$ to $|x_0\rangle$, and $|z_1\rangle$ to $|z_0\rangle$. Registers now hold $|x_1\rangle |x_0 + x_1\rangle |z_1\rangle |z_0 + z_1\rangle$.
5. Apply PhaseProduct($2^{n/2}\phi, |x_0 + x_1\rangle, |z_0 + z_1\rangle$).
6. Subtract $|x_1\rangle$, $|z_1\rangle$ to return to registers to $|x_1\rangle |x_0\rangle |z_1\rangle |z_0\rangle$.

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Karatsuba is Toom-Cook with $k = 2$

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